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On the Quasi-Static Evolution of Nonequilibrium Steady States

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Abstract. The quasi-static evolution of steady states *far from equilibrium* is investigated from the point of view of quantum statistical mechanics. As a concrete example of a thermodynamic system, a two-level *quantum dot* coupled to several reservoirs of free fermions at different temperatures is considered. A novel adiabatic theorem for unbounded and nonnormal generators of evolution is proven and applied to study the quasi-static evolution of the nonequilibrium steady state (NESS) of the coupled system.

1. Introduction

Recently, there has been substantial progress in understanding and rigorously proving the asymptotic convergence (as time $t \to \infty$) of a state of a thermodynamic system, say one composed of a finitely extended system coupled to one or more thermal reservoir, to a steady state, both in equilibrium [6, 8, 9, 14, 15, 19, 20] and far from equilibrium [10, 16, 21, 22, 26, 27] from the point of view of quantum statistical mechanics. After the state of a certain thermodynamic system reaches a steady state, it is natural to ask how the state will evolve if the system is perturbed slowly over time scales that are large compared to a generic relaxation time of the system, and how much the state of the system will be close to the instantaneous (non)equilibrium steady state.

This question was first addressed in [1], where the *isothermal theorem*, an adiabatic theorem for states close to thermal equilibrium, has been proven, and applications of this theorem to reversible isothermal processes have been discussed. Here, we pursue this question further by investigating the quasi-static evolution of states *far from equilibrium* from the point of view of quantum statistical mechanics.

According to the spectral approach to nonequilibrium steady states (NESS), the latter corresponds to a zero-energy resonance of the (adjoint of the) C-Liouvillean; (see [16, 21, 22]). Since the C-Liouvillean is generally nonnormal and unbounded, we prove an adiabatic theorem for generators of evolution that are not necessarily bounded or normal. This theorem can be extended to study the adiabatic evolution of quantum resonances. [2]

As a concrete example of a thermodynamic system, we consider a system composed of a two-level quantum system coupled to several fermionic reservoirs at different temperatures (for example, a *quantum dot* coupled to electrons in several metals). We apply the general adiabatic theorem to study the adiabatic evolution of the NESS for this system. The main ingredients of our analysis are an adiabatic theorem for nonnormal and unbounded generators of evolution, a concrete representation of the fermionic reservoirs (Araki–Wyss representation [4]), the spectral approach to NESS using C-Liouvilleans, and complex deformation techniques as developed in [12, 14–16].

The organization of this paper is as follows. In Section 2, we state and prove a general adiabatic theorem (Theorem 2.2). This is the key result of this section, which we apply in the subsequent sections to study the quasi-static evolution of nonequilibrium steady states. In Section 3, we discuss the concrete physical model we consider: a two level quantum system coupled to several fermionic reservoirs at different temperatures.[1] In Section 4, we study the C-Liouvillean corresponding to the coupled system using complex deformation techniques (Theorem 4.3), and recall the relationship between the NESS and a zero-energy resonance of the C-Liouvillean (Corollary 4.4). In Section 5, we apply Theorem 2.2 to study the adiabatic evolution of the NESS of the coupled system. The main result of this section is Theorem 5.1. We also remark on the strict positivity of entropy production in the quasi-static evolution of NESS, and on a concrete example of the isothermal theorem [1]. Some technical details and proofs are collected in an Appendix.

2. A general adiabatic theorem

So far, adiabatic theorems that are considered in the literature deal with generators of evolution which are self-adjoint; (see for example [5]). This is expected, since the generator of dynamics in quantum mechanics, the Hamiltonian, is self-adjoint. However, for systems out of equilibrium, a generally nonnormal and unbounded operator, the so called C-Liouvillean, can be used to generate an equivalent dynamics on a suitable Banach space. Since we are interested in studying the quasi-static evolution of NESS, it is useful to prove an adiabatic theorem for nonnormal generators of time evolution. This is what is done in this section.

Consider a family of closed operators $\{A(t)\}, t \in \mathbf{R}^+$, acting on a Hilbert space $\mathcal{H}$. We make the following assumptions on $A(t)$ in order to prove the existence of a time evolution and to prove an adiabatic theorem. *All* of these assumptions will be verified in the applications which are considered in the subsequent sections.

(A1) $A(t)$ is a generator of a contraction semi-group for all $t \in \mathbf{R}^+$.

(A2) $A(t)$ have a common dense domain $\mathcal{D} \subset \mathcal{H}$ for all $t \in \mathbf{R}^+$.

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[1] The analysis can be directly generalized to the case when the small system is coupled to several bosonic reservoirs by using methods developed in [21, 22].

(A3) For $z \in \rho(A(t))$, the resolvent set of $A(t)$, let $R(z,t) := (z - A(t))^{-1}$. Assume that $R(-1,t)$ is bounded and differentiable as a bounded operator on $\mathcal{H}$, and that $A(t)\dot{R}(-1,t)$ is bounded, where the $(\dot{})$ stands for differentiation with respect to t. Moreover, assume that for every $\epsilon > 0$, $-\epsilon \in \rho(A(t))$.

Let $U(t)$ be the propagator that satisfies

$$\partial_t U(t)\psi = -A(t)U(t)\psi\,, \quad U(t=0) = 1\,, \tag{1}$$

for $t \geq 0$; $\psi \in \mathcal{D}$. We have the following result.

Lemma 2.1. *Suppose that assumptions* (A1)–(A3) *hold. Then the propagator* $U(t)$ *satisfying* (1) *exists and is unique, and* $\|U(t)\psi\| \leq \|\psi\|$, *for* $\psi \in \mathcal{D}$.

The result of Lemma 2.1 is standard, and it follows from assumptions (A1)–(A3) above and Theorem X.70 in [24].[2]

Assume that $A(t) \equiv A(0)$ for $t \leq 0$, and that it is perturbed *slowly* over a time τ such that $A^{(\tau)}(t) \equiv A(s)$, where $s := t/\tau \in [0,1]$ is the rescaled time. The following additional two assumptions are needed to prove an adiabatic theorem.

(A4) The eigenvalue $\lambda(s) \in \sigma(A(s))$ is isolated and simple, such that

$$dist\Big(\lambda(s), \sigma\big(A(s)\big)\backslash\{\lambda(s)\}\Big) > d\,,$$

where $d > 0$ is a constant independent of $s \in [0,1]$, and $\lambda(s)$ is continuously differentiable in $s \in [0,1]$.

(A5) The projection onto $\lambda(s)$,

$$P_\lambda(s) := \frac{1}{2\pi i}\oint_{\gamma_\lambda(s)} R(z,s)dz\,, \tag{2}$$

where $\gamma_\lambda(s)$ is a contour enclosing $\lambda(s)$ only, is twice differentiable as a bounded operator.

Note that, since $\lambda(s)$ is simple, the resolvent of $A(s)$ in a neighborhood $\mathcal{N}$ of $\lambda(s)$ contained in a ball $\mathbf{B}(\lambda(s), r)$ centered at $\lambda(s)$ with radius $r < d$ is

$$R(z,s) = \frac{P_\lambda(s)}{z - \lambda(s)} + R_{analytic}(z,s)\,, \tag{3}$$

where $R_{analytic}(z,s)$ is analytic in $\mathcal{N}$. We recall some useful properties of the resolvent and the spectral projection $P_\lambda(s)$; (see [17]).

[2] Choose $\eta > 0$ and let $\widetilde{U}(t)$ be the propagator generated by $\widetilde{A}(t) := A(t) + \eta$. It follows from (A1) that $\widetilde{A}(t)$ is a generator of a contraction semigroup. Furthermore, for $t, t' \in \mathbf{R}^+$, $\widetilde{A}(t')\widetilde{A}(t)^{-1}$ is bounded due to the closed graph theorem and (A2) (see [23]). Moreover, for small $|t - t'|$, $||(t' - t)(\widetilde{A}(t')\widetilde{A}(t)^{-1} - \mathbf{1})|| = ||\widetilde{A}(t)\dot{\widetilde{A}}^{-1}(t)|| + o(|t - t'|)$, which is bounded due to (A3). By Theorem X.70 in [24] (or Theorem 2, Chapter XIV in [28], Section 4), this implies, together with (A1) and (A2), that $\widetilde{U}(t)$ exists and is unique. In particular, $\|\widetilde{U}(t)\psi\| \leq 1$ uniformly in $t \geq 0$ (for $\|\psi\| = 1$). We also have $\|U(t)\| = e^{\eta t}\|\widetilde{U}(t)\|$. Taking the limit $\eta \to 0$ gives $\|U(t)\psi\| \leq 1$.

(i) It follows by direct application of the contour integration formula that

$$\big(P_\lambda(s)\big)^2 = P_\lambda(s)\ , \tag{4}$$

and hence

$$P_\lambda(s)\dot{P}_\lambda(s)P_\lambda(s) = 0\,. \tag{5}$$

(ii)

$$A(s)P_\lambda(s) = P_\lambda(s)A(s) = \lambda(s)P_\lambda(s)\,. \tag{6}$$

Proof.

$$\begin{aligned} A(s)P_\lambda(s) &= \frac{1}{2\pi i}\oint_{\gamma_\lambda(s)} \big(A(s) - z + z\big)\big(z - A(s)\big)^{-1} dz \\ &= \frac{1}{2\pi i}\left\{ -\oint_{\gamma_\lambda(s)} dz + \oint_{\gamma_\lambda(s)} \left(\frac{zP_\lambda(s)}{z-\lambda(s)} + zR_{analytic}\right) dz \right\} \\ &= \lambda(s)P_\lambda(s)\,, \end{aligned}$$

and similarly, $P_\lambda(s)A(s) = \lambda(s)P_\lambda(s)$. □

(iii) It follows from (3) and (A4) that, for $\eta \in \mathbf{C}$ and $d/2 \leq |\eta| < d$, there exists a constant $C < \infty$, independent of η, such that

$$\|R(\lambda(s)+\eta, s)\| < C\,, \tag{7}$$

uniformly in $s \in [0,1]$. Moreover, since $(\lambda(s)+\eta) \in \rho(A(s))$, it follows by the spectral mapping theorem (see for example [28], Chapter VIII, Section 7) and (A3) that $R(\lambda(s)+\eta, s)$ is differentiable as a bounded operator.[3]

We now discuss our general adiabatic theorem. Let $U_\tau(s,s')$ be the propagator satisfying

$$\partial_s U_\tau(s,s') = -\tau A(s)U_\tau(s,s')\,, \quad U_\tau(s,s) = 1\,, \tag{8}$$

for $s \geq s'$. Moreover, define the generator of the *adiabatic time evolution*,

$$A_a(s) := A(s) - \frac{1}{\tau}\big[\dot{P}_\lambda(s), P_\lambda(s)\big]\,, \tag{9}$$

with the corresponding propagator $U_a(s,s')$ which satisfies

$$\partial_s U_a(s,s') = -\tau A_a(s)U_a(s,s')\,; \quad U_a(s,s) = 1\,, \tag{10}$$

for $s \geq s'$.

By Lemma 2.1 and (A1)–(A3) and (A5), both propagators $U_\tau(s,s')$ and $U_a(s,s')$ exist and are unique, and $\|U_\tau(s,s')\|, \|U_a(s,s')\| < C$ for $s \geq s'$, where C is a finite constant independent of $s, s' \in [0,1]$. We are in a position to state our adiabatic theorem.

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[3] We know that, for $z, \omega \in \rho(A)$,

$$(z-A)^{-1} = (1 + (z-\omega)(\omega - A)^{-1})^{-1}(\omega - A)^{-1}\,.$$

In particular, choose $z = \lambda(s)+\eta$ and $\omega = -1$. Differentiability of $R(\lambda(s)+\eta)$ as a bounded operator follows from the latter identity and assumption (A3).

Theorem 2.2 (A general adiabatic theorem). *Assume* (A1)–(A5). *Then the following holds.*

(i)
$$P_\lambda(s)U_a(s,0) = U_a(s,0)P_\lambda(0)\,, \tag{11}$$
for $s \geq 0$ (the intertwining property).

(ii) *There is a finite constant C such that*
$$\sup_{s\in[0,1]} \|U_\tau(s,0) - U_a(s,0)\| \leq \frac{C}{1+\tau}\,,$$
for $\tau > 0$. In particular,
$$\sup_{s\in[0,1]} \|U_\tau(s,0) - U_a(s,0)\| = O(\tau^{-1})\,,$$
as $\tau \to \infty$.

Proof. (i) Equality holds trivially for $s = 0$, since $U_a(s,s) = 1$. Let
$$h(s,s') := U_a(s,s')P_\lambda(s')U_a(s',0)\,, \tag{12}$$
for $0 \leq s' \leq s$.

Using (6), (10) , the definition of $A_a(s)$ and the fact that $\dot{P}_\lambda(s)P_\lambda(s) + P_\lambda(s)\dot{P}_\lambda(s) = \dot{P}_\lambda(s)$, it follows that
$$\begin{aligned}
\partial_{s'} h(s,s') &= \partial_{s'}\big(U_a(s,s')P_\lambda(s')U_a(s',0)\big) \\
&= \tau U_a(s,s')\big\{A_a(s')P_\lambda(s') - P_\lambda(s')A_a(s')\big\}U_a(s',0) \\
&\quad + U_a(s,s')\dot{P}_\lambda(s')U_a(s',0) \\
&= U_a(s,s')\big\{ - \dot{P}_\lambda(s')P_\lambda(s') \\
&\quad - P_\lambda(s')\dot{P}_\lambda(s') + \dot{P}_\lambda(s')\big\}U_a(s',0) \\
&= 0\,.
\end{aligned}$$

Therefore,
$$h(s,s') \equiv h(s)\,.$$

In particular,
$$h(s,s) = h(s,0)\,,$$
which implies claim (i).

(ii) Consider $\psi \in \mathcal{D}$, where the dense domain $\mathcal{D}$ appears in assumption (A2). We are interested in estimating the norm of the difference $(U_\tau(s,0) - U_a(s,0))\psi$ as $\tau \to \infty$. Using (8), (10) and the Duhamel formula,
$$\begin{aligned}
\big(U_\tau(s,0) - U_a(s,0)\big)\psi &= -\int_0^s ds'\partial_{s'}\big(U_\tau(s,s')U_a(s',0)\big)\psi \qquad (13)\\
&= \int_0^s ds'\Big(U_\tau(s,s')\big[\dot{P}_\lambda(s'), P_\lambda(s')\big]U_a(s',0)\Big)\psi\,. \qquad (14)
\end{aligned}$$

Let

$$X(s) := \frac{1}{2\pi i}\oint_{\gamma_\lambda(s)} dz R(z,s)\dot{P}_\lambda(s)R(z,s)\,, \tag{15}$$

where $\gamma_\lambda(s)$ is a contour of radius $d/2$ centered at $\lambda(s)$, and where d appears in (A4). Then

$$\begin{aligned}\big[X(s), A(s)\big] &= \frac{1}{2\pi i}\oint_{\gamma_\lambda(s)} dz\big[z - A(s), R(z,s)\dot{P}_\lambda(s)R(z,s)\big] \\ &= \dot{P}_\lambda(s)P_\lambda(s) - P_\lambda(s)\dot{P}_\lambda(s) = \big[\dot{P}_\lambda(s), P_\lambda(s)\big]\,. \end{aligned}\tag{16}$$

Assumptions (A3),(A4) and the spectral mapping theorem imply that, for $z \in \gamma_\lambda(s) \subset \rho(A(s)), R(z,s)$ is differentiable as a bounded operator. Together with (A5), this implies that,

$$\|X(s)\| < C_1\,, \tag{17}$$

$$\|\dot{X}(s)\| < C_2\,, \tag{18}$$

where C_1 and C_2 are finite constants independent of $s \in [0,1]$. Moreover,

$$\begin{aligned}U_\tau(s,s')\big[X(s'), A(s')\big]U_a(s',0) = \frac{1}{\tau}\Big\{ &-\partial_{s'}U_\tau(s,s')X(s')U_a(s',0) \\ &+ U_\tau(s,s')\Big(X(s')\big[\dot{P}_\lambda(s'), P_\lambda(s')\big]\Big)U_a(s',0) + U_\tau(s,s')\dot{X}(s')U_a(s',0)\Big\}\,.\end{aligned}$$

Together with (16), one may write the integrand in (13) as a total derivative plus a remainder term. Using the fact that $\mathcal{D}$ is dense in $\mathcal{H}$ and (A5),

$$\|U_\tau(s,0) - U_a(s,0)\| \le \frac{1}{\tau}\sup_{s\in[0,1]}\big[C_1'\|X(s)\| + C_2'\|\dot{X}(s)\|\big]\,, \tag{19}$$

where $C_i', i = 1,2$ are finite constants independent of $s \in [0,1]$.

Together with (17) and (18), this implies

$$\sup_{s\in[0,1]}\|U_\tau(s,0) - U_a(s,0)\| \le \frac{C}{1+\tau}\,, \tag{20}$$

for $\tau > 0$, where C is a finite positive constant. □

Next, we discuss a concrete model of a thermodynamic system to be studied subsequently.

3. The model

As an example, we consider a two-level quantum system Σ coupled to n reservoirs, $\mathcal{R}_1, \ldots, \mathcal{R}_n, n \ge 2$, of free fermions in thermal equilibrium at inverse temperatures $\beta_1, \ldots, \beta_n$.

The small system

The kinematical algebra of Σ is $\mathcal{O}^{\Sigma} = \mathcal{M}(\mathbf{C}^2)$, the algebra of complex 2×2 matrices over the Hilbert space $\mathcal{H}^{\Sigma} = \mathbf{C}^2$. Its Hamiltonian is given by $H^{\Sigma} = \omega_0 \sigma_3$, where $\sigma_i, i = 1, 2, 3$, are the Pauli matrices. When the system Σ is not coupled to the reservoirs, its dynamics in the Heisenberg picture is given by

$$\alpha_{\Sigma}^{t,s}(a) := e^{iH^{\Sigma}(t-s)} a e^{-iH^{\Sigma}(t-s)}, \tag{21}$$

for $a \in \mathcal{O}^{\Sigma}$.

A physical state of the small system, ω^{Σ}, is described by a density matrix ρ_{Σ}. We assume that $\rho_{\Sigma} > 0$, i.e., ω^{Σ} is faithful. The operator $\kappa_{\Sigma} = \rho_{\Sigma}^{1/2}$ belongs to the space of Hilbert–Schmidt operators, whyich is isomorphic to $\mathcal{H}^{\Sigma} \otimes \mathcal{H}^{\Sigma}$. Two commuting representations of $\mathcal{O}^{\Sigma}$ on $\mathcal{H}^{\Sigma} \otimes \mathcal{H}^{\Sigma}$ are given by

$$\pi_{\Sigma}(a) := a \otimes \mathbf{1}^{\Sigma}, \tag{22}$$

$$\pi_{\Sigma}^{\#}(a) := \mathbf{1}^{\Sigma} \otimes C^{\Sigma} a C^{\Sigma}, \tag{23}$$

where C^{Σ} is an antiunitary involution on $\mathcal{H}^{\Sigma}$ corresponding to complex conjugation in the basis of the eigenvectors of H^{Σ}; (see for example [6]).

The generator of the free dynamics on the Hilbert space $\mathcal{H}^{\Sigma} \otimes \mathcal{H}^{\Sigma}$ is the standard Liouvillean

$$\mathcal{L}^{\Sigma} = H^{\Sigma} \otimes \mathbf{1}^{\Sigma} - \mathbf{1}^{\Sigma} \otimes H^{\Sigma}. \tag{24}$$

The spectrum of $\mathcal{L}^{\Sigma}$ is $\sigma(\mathcal{L}^{\Sigma}) = \{-2\omega_0, 0, 2\omega_0\}$, with double degeneracy at zero.

Let ω^{Σ} be the initial state of the small system Σ, with corresponding vector $\Omega^{\Sigma} \in \mathcal{H}^{\Sigma} \otimes \mathcal{H}^{\Sigma}$. The modular operator associated with ω^{Σ} is $\Delta^{\Sigma} = \omega^{\Sigma} \otimes \overline{\omega^{\Sigma}}^{-1}$, and the modular conjugation operator, J^{Σ}, is given by

$$J^{\Sigma}(\phi \otimes \psi) = \overline{\psi} \otimes \overline{\phi},$$

for $\phi, \psi \in \mathcal{H}^{\Sigma}$. If ω^{Σ} corresponds to the trace state, then $\Delta^{\Sigma} = \mathbf{1}^{\Sigma} \otimes \mathbf{1}^{\Sigma}$.

The reservoirs

Each thermal reservoir is formed of free fermions. It is infinitely extended and *dispersive*. We assume that the Hilbert space of a single fermion is $\mathbf{h} = L^2(\mathbf{R}^+; \mathcal{B})$, where $\mathcal{B}$ is an auxiliary Hilbert space, and $m(u)du$ is a measure on $\mathbf{R}^+$. We also assume that the single-fermion Hamiltonian, h, corresponds to the operator of multiplication by $u \in \mathbf{R}^+$. For instance, for reservoirs formed of nonrelativistic fermions in $\mathbf{R}^3$, the auxiliary Hilbert space $\mathcal{B}$ is $L^2(S^2, d\sigma)$, where S^2 is the unit sphere in $\mathbf{R}^3, d\sigma$ is the uniform measure on S^2, and $u = |\vec{k}|^2$, where $\vec{k} \in \mathbf{R}^3$ is the particle's momentum. In the latter case, the measure on $\mathbf{R}^+$ is chosen to be $m(u)du = \frac{1}{2}\sqrt{u}du$.

Let b and b^* be the annihilation-and creation operators on the Fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$. They satisfy the usual canonical anticommutation relation (CAR)

$$\{b^{\#}(f), b^{\#}(g)\} = 0\,, \tag{25}$$

$$\{b(f), b^*(g)\} = (f,g)\mathbf{1}\,, \tag{26}$$

where $b^{\#}$ stands for b or b^*, $f,g \in L^2(\mathbf{R}^+;\mathcal{B})$, and $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbf{R}^+;\mathcal{B})$. Moreover, let $\Omega^{\mathcal{R}}$ denote the vacuum state in $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$.

The kinematical algebra, $\mathcal{O}^{\mathcal{R}_i}$, of the i^{th} reservoir $\mathcal{R}_i, i = 1,\dots,n$, is generated by $b_i^{\#}$ and the identity $\mathbf{1}^{\mathcal{R}_i}$. The free dynamics of each reservoir (before the systems are coupled) is given by

$$\alpha^{t,s}_{\mathcal{R}_i}\big(b_i^{\#}(f)\big) = b_i^{\#}\big(e^{i(t-s)u} f\big)\,, \tag{27}$$

for $i = 1,\dots,n, f \in L^2(\mathbf{R}^+;\mathcal{B})$.

The $(\alpha_{\mathcal{R}_i}, \beta_i)$-KMS state, $\omega^{\mathcal{R}_i}$, of each reservoir $\mathcal{R}_i, i = 1,\dots,n$, at inverse temperature β_i, is the gauge invariant, quasi-free state uniquely determined by the two-point function

$$\omega^{\mathcal{R}_i}\big(b_i^*(f) b_i(f)\big) = \big(f, \rho_{\beta_i}(\cdot) f\big)\,, \tag{28}$$

where $\rho_{\beta_i}(u) := \frac{1}{e^{\beta_i u}+1}$.

Next, we introduce $\mathcal{F}_i^{AW} := \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+;\mathcal{B})) \otimes \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+;\mathcal{B}))$, the GNS Hilbert space for the Araki–Wyss representation of each fermionic reservoir $\mathcal{R}_i$ associated with the state $\omega^{\mathcal{R}_i}$, [4]. Denote by $\Omega^{\mathcal{R}_i}$ the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+;\mathcal{B}))$, with $b_i \Omega^{\mathcal{R}_i} = 0$. The Araki–Wyss representation, π_i, of the kinematical algebra $\mathcal{O}^{\mathcal{R}_i}, i = 1,\dots,n$, on $\mathcal{F}_i^{AW}$ is given by

$$\begin{aligned}
\pi_i\big(b_i(f)\big) &:= b_i\big(\sqrt{1-\rho_{\beta_i}}\, f\big) \otimes \mathbf{1}^{\mathcal{R}_i} + (-1)^{N_i} \otimes b_i^*\big(\sqrt{\rho_{\beta_i}}\, \overline{f}\,\big)\,, \qquad (29)\\
\pi_i^{\#}\big(b_i(f)\big) &:= b_i^*\big(\sqrt{\rho_{\beta_i}} f\big)(-1)^{N_i} \otimes (-1)^{N_i} + \mathbf{1}^{\mathcal{R}_i} \otimes (-1)^{N_i} b_i\big(\sqrt{1-\rho_{\beta_i}}\, \overline{f}\,\big)\,,
\end{aligned}$$

where $N_i = d\Gamma_i(1)$ is the particle number operator for reservoir $\mathcal{R}_i$. Furthermore, $\Omega^{\mathcal{R}_i} \otimes \Omega^{\mathcal{R}_i} \in \mathcal{F}_i^{AW}$ corresponds to the equilibrium KMS state $\omega^{\mathcal{R}_i}$ of reservoir $\mathcal{R}_i$.

The free dynamics on the GNS Hilbert space $\mathcal{F}_i^{AW}$ of each reservoir $\mathcal{R}_i$ is generated by the standard Liouvillean $\mathcal{L}^{\mathcal{R}_i}$. The modular operator associated with $(\mathcal{O}^{\mathcal{R}_i}, \omega^{\mathcal{R}_i})$ is given by

$$\Delta^{\mathcal{R}_i} = e^{-\beta_i \mathcal{L}^{\mathcal{R}_i}}\,,$$

and the modular conjugation is given by

$$J^{\mathcal{R}_i}(\Psi \otimes \Phi) = (-1)^{N_i(N_i-1)/2}\overline{\Phi} \otimes (-1)^{N_i(N_i-1)/2}\overline{\Psi}\,,$$

for $\Psi, \Phi \in \mathcal{F}_i^{AW}$; (see, for example, [7]).

In order to apply the complex translation method developed in [12, 14–16], we map $\mathcal{F}_i^{AW} = \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+;\mathcal{B})) \otimes \mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}^+;\mathcal{B}))$ to $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$ as done in [16]; (using the isomorphism between $L^2(\mathbf{R}^+;\mathcal{B}) \oplus L^2(\mathbf{R}^+;\mathcal{B})$ and $L^2(\mathbf{R};\mathcal{B})$, the latter having measure du on $\mathbf{R}$). To every $f \in L^2(\mathbf{R}^+;\mathcal{B})$, we associate functions $f_\beta, f_\beta^\# \in L^2(\mathbf{R};\mathcal{B})$, with measure du on $\mathbf{R}$, by setting

$$f_\beta(u,\sigma) := \begin{cases} \sqrt{m(u)}\sqrt{1-\rho_\beta(u)}f(u,\sigma)\,, & u \geq 0 \\ \sqrt{m(-u)}\sqrt{\rho_\beta(-u)}\;\overline{f}(-u,\sigma)\,, & u < 0 \end{cases}, \tag{30}$$

and

$$\begin{aligned} f_\beta^\#(u,\sigma) &:= \begin{cases} \sqrt{m(u)}i\sqrt{\rho_\beta(u)}f(u,\sigma)\,, & u \geq 0 \\ \sqrt{m(-u)}i\sqrt{1-\rho_\beta(-u)}\;\overline{f}(-u,\sigma)\,, & u < 0 \end{cases} \\ &= i\overline{f}_\beta(-u,\sigma)\,, \end{aligned} \tag{31}$$

where $m(u)du$ is the measure on $\mathbf{R}^+$, see (29).

Let a_i and a_i^* be the annihilation and creation operators on $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$. Then

$$\pi_i\big(b_i(f)+b_i^*(f)\big) \to a_i(f_{\beta_i}) + a_i^*(f_{\beta_i})\,, \tag{32}$$

$$\pi_i^\#\big(b_i(f)+b_i^*(f)\big) \to i(-1)^{N_i}\big[a_i(f_{\beta_i}^\#) + a_i^*(f_{\beta_i}^\#)\big]\,; \tag{33}$$

$$\Omega^{\mathcal{R}_i}\otimes\Omega^{\mathcal{R}_i} \to \widetilde{\Omega}^{\mathcal{R}_i}\,, \tag{34}$$

where $\widetilde{\Omega}^{\mathcal{R}_i}$ is the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$.[4]

Moreover, the free Liouvillean on $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$ for the reservoir $\mathcal{R}_i$ is mapped to

$$\mathcal{L}^{\mathcal{R}_i} = d\Gamma_i(u)\,, \tag{35}$$

where $u \in \mathbf{R}$.

The coupled system

The kinematical algebra of the total system, $\Sigma \vee \mathcal{R}_1 \vee \cdots \vee \mathcal{R}_n$, is given by

$$\mathcal{O} = \mathcal{O}^\Sigma \otimes \mathcal{O}^{\mathcal{R}_1} \otimes \cdots \otimes \mathcal{O}^{\mathcal{R}_n}\,, \tag{36}$$

and the Heisenberg-picture dynamics of the uncoupled system is given by

$$\alpha_0 = \alpha_\Sigma \otimes \alpha_{\mathcal{R}_1} \otimes \cdots \otimes \alpha_{\mathcal{R}_n}\,. \tag{37}$$

The representation of $\mathcal{O}$ on $\mathcal{H} := \mathcal{H}^\Sigma \otimes \mathcal{H}^\Sigma \otimes \mathcal{F}^{\mathcal{R}_1}(L^2(\mathbf{R};\mathcal{B})) \otimes \cdots \otimes \mathcal{F}^{\mathcal{R}_n}$ $(L^2(\mathbf{R};\mathcal{B}))$, determined by the initial state

$$\omega = \omega^\Sigma \otimes \omega^{\mathcal{R}_1} \otimes \cdots \otimes \omega^{\mathcal{R}_n} \tag{38}$$

by the GNS construction, is given by

$$\pi = \pi_\Sigma \otimes \pi_1 \otimes \cdots \otimes \pi_n\,, \tag{39}$$

and an anti-representation commuting with π by

$$\pi^\# = \pi_\Sigma^\# \otimes \pi_1^\# \otimes \cdots \otimes \pi_n^\#\,. \tag{40}$$

[4]For a discussion of this map, see Theorem 3.3 in [16]; (see also the Appendix).

Moreover, let $\Omega := \Omega^{\Sigma} \otimes \widetilde{\Omega}^{\mathcal{R}_1} \otimes \cdots \otimes \widetilde{\Omega}^{\mathcal{R}_n}$ denote the vector in $\mathcal{H}$ corresponding to the state ω. Denote the double commutant of $\pi(\mathcal{O})$ by $\mathcal{M} := \pi(\mathcal{O})''$, which is the smallest von Neumann algebra containing $\pi(\mathcal{O})$.

For $a \in \mathcal{O}$, we abbreviate $\pi(a)$ by a whenever there is no danger of confusion. The modular operator of the total system is

$$\Delta = \Delta^{\Sigma} \otimes \Delta^{\mathcal{R}_1} \otimes \cdots \otimes \Delta^{\mathcal{R}_n},$$

and the modular conjugation is

$$J = J^{\Sigma} \otimes J^{\mathcal{R}_1} \otimes \cdots \otimes J^{\mathcal{R}_n}\,.$$

According to Tomita–Takesaki theory,

$$J\mathcal{M}J = \mathcal{M}'\,, \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}\,,$$

for $t \in \mathbf{R}$; (see for example [7]). Furthermore, for $a \in \mathcal{M}$,

$$J\Delta^{1/2}a\Omega = a^*\Omega\,. \tag{41}$$

The Liouvillean of the total uncoupled system is given by

$$\mathcal{L}_0 = \mathcal{L}^{\Sigma} + \sum_{i=1}^{n} \mathcal{L}^{\mathcal{R}_i}\,. \tag{42}$$

This defines a selfadjoint operator on $\mathcal{H}$.

The system Σ is coupled to the reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_n$, through an interaction $gV(t)$, where $V(t) \in \mathcal{O}$ is given by

$$V(t) = \sum_{i=1}^{n} \left\{ \sigma_1 \otimes \left[b_i\big(f_i(t)\big) + b_i^*\big(f_i(t)\big) \right] \right\}, \tag{43}$$

$\sigma_i, i = 1, 2, 3$, are the Pauli matrices, and $f_i \in L^2(\mathbf{R}^+; \mathcal{B}), i = 1, \ldots, n$, are the form factors.

The *standard* Liouvillean of the interacting system acting on the GNS Hilbert space $\mathcal{H}$ is given by

$$\mathcal{L}_g(t) = \mathcal{L}_0 + gI(t)\,, \tag{44}$$

where the unperturbed Liouvillean is defined in (42), and the interaction Liouvillean determined by the operator $V(t)$ is given by

$$\begin{aligned} I(t) &= \big\{V(t) - JV(t)J\big\} \\ &= \sum_{i=1}^{n} \Big\{ \sigma_1 \otimes \mathbf{1}^{\Sigma} \otimes \Big[a_i^*\big(f_{i,\beta_i}(t)\big) + a_i\big(f_{i,\beta_i}(t)\big) \Big] \\ &\quad - i\mathbf{1}^{\Sigma} \otimes \sigma_1 \otimes (-1)^{N_i} \Big[a_i^*\big(f^{\#}_{i,\beta_i}(t)\big) + a_i\big(f^{\#}_{i,\beta_i}(t)\big) \Big] \Big\}, \end{aligned} \tag{45}$$

where a_i, a_i^* are the annihilation and creation operators on the fermionic Fock space $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R}; \mathcal{B}))$. Note that since the perturbation is bounded, the domain of $\mathcal{L}_g(t)$ is $\mathcal{D}(\mathcal{L}_g(t)) = \mathcal{D}(\mathcal{L}_0)$.

Let $\overline{U}_g$ be the propagator generated by the standard Liouvillean. It satisfies

$$\partial_t \overline{U}_g(t,t') = -i\mathcal{L}_g(t)\overline{U}_g(t,t')\,; \quad \overline{U}_g(t,t) = 1\,, \tag{46}$$

for $t \geq t'$. The Heisenberg-picture evolution is given by

$$\alpha_g^{t,t'}(a) = \overline{U}_g^*(t,t')a\overline{U}_g(t,t')\,, \tag{47}$$

for $a \in \mathcal{O}$.

Generally, the kernel of $\mathcal{L}_g(t), Ker\ \mathcal{L}_g$, is expected to be empty when at least two of the reservoirs have different temperatures.[5] This motivates introducing the so called C-Liouvillean, L_g , which generates an equivalent dynamics on a suitable Banach space contained in $\mathcal{H}$ (isomorphic to $\mathcal{O}$) and which, *by construction*, has a non-trivial kernel.

Consider the Banach space

$$\mathcal{C}(\mathcal{O},\Omega) := \{a\Omega : a \in \mathcal{O}\}\,,$$

with norm $\|a\Omega\|_\infty = \|a\|$. Since Ω is separating for $\mathcal{O}$, the norm $\|a\Omega\|_\infty$ is well-defined, and since Ω is cyclic for $\mathcal{O}$, $\mathcal{C}(\mathcal{O},\Omega)$ is dense in $\mathcal{H}$.

Let $U_g(t,t')$ be the propagator given by

$$\alpha_g^{t,t'}(a)\Omega = U_g(t,t')a\Omega\,, \tag{48}$$

and

$$U_g(t',t)\Omega = \Omega\,. \tag{49}$$

Moreover, let $L_g(t)$ be its generator, i.e.,

$$\partial_t U_g(t,t') = iU_g(t,t')L_g(t) \quad \text{with} \quad U_g(t,t) = 1\,. \tag{50}$$

Differentiating (48) with respect to t, setting $t = t'$, and using (50), (47) and (41), one obtains

$$\begin{aligned}\Big[\big(\mathcal{L}_0 + gV(t)\big)a - a\big(\mathcal{L}_0 + gV(t)\big)\Big]\Omega &= \Big[\big(\mathcal{L}_0 + gV(t)\big)a - \big(V(t)a^*\big)^*\Big]\Omega \\ &= \big(\mathcal{L}_0 + gV(t) - gJ\Delta^{1/2}V(t)\Delta^{-1/2}J\big)a\Omega \\ &\equiv L_g(t)a\Omega\ .\end{aligned}$$

Hence, the C-Liouvillean is given by

$$L_g(t) := \mathcal{L}_0 + gV(t) - gJ\Delta^{1/2}V(t)\Delta^{-1/2}J\,. \tag{51}$$

Note that, *by construction*,

$$L_g(t)\Omega = 0\,,$$

for all $t \in \mathbf{R}$.

Next, we discuss the assumptions on the interaction. For $\delta > 0$, we define the strips in the complex plane

$$I(\delta) := \{z \in \mathbf{C} : |Imz| < \delta\}$$

[5] This is consistent with the fact that the coupled system is not expected to possess the property of return to equilibrium if the reservoirs have different temperatures (or chemical potentials). One can verify that, indeed, this is the case when assumptions (B1) and (B2), below, are satisfied; (see [16, 21, 22]).

and

$$I^-(\delta) := \{z \in \mathbf{C} : -\delta < Imz < 0\} . \tag{52}$$

Moreover, for every function $f \in L^2(\mathbf{R}^+; \mathcal{B})$, we define a function $\widetilde{f}$ by setting

$$\widetilde{f}(u,\sigma) := \begin{cases} \sqrt{m(u)} f(u,\sigma) , & u \geq 0 \\ \sqrt{m(|u|)}\, \overline{f}(|u|,\sigma) , & u < 0 \end{cases} , \tag{53}$$

where $m(u)du$ is the measure on $\mathbf{R}^+$. Denote by $H^2(\delta, \mathcal{B})$ the Hardy class of analytic functions

$$h : I(\delta) \to \mathcal{B} ,$$

with

$$\|h\|_{H^2(\delta,\mathcal{B})} := \sup_{|\theta|<\delta} \int_{\mathbf{R}} \|h(u+i\theta)\|^2_{\mathcal{B}} du < \infty .$$

We require the following basic assumptions on the interaction term.

(B1) *Fermi Golden Rule.*

Assume that

$$\sum_{i=1}^{n} \|\widetilde{f}_i(2\omega_0, t)\|_{\mathcal{B}} > 0 , \tag{54}$$

for almost all $t \in \mathbf{R}$, which is another way of saying that the small system is coupled to at least one reservoir, to second order in perturbation theory.

(B2) *Regularity of the form factors.*

Assume that $\exists \delta > 0$, independent of t and $i = 1, \ldots, n$, such that

$$e^{-\beta_i u/2} \widetilde{f}_i(u,t) \in H^2(\delta, \mathcal{B}) , \tag{55}$$

the Hardy class of analytic functions. This implies that the mapping

$$\mathbf{R} \ni r \to \Delta^{ir} V(t) \Delta^{-ir} \in \mathcal{M} , \tag{56}$$

(where $\Delta = \Delta^{\Sigma} \otimes \Delta^{\mathcal{R}_1} \otimes \cdots \otimes \Delta^{\mathcal{R}_n}$ is the modular operator of the coupled system, and $\mathcal{M} = \pi(\mathcal{O})''$,) has an analytic continuation to the strip $I(1/2) = \{z \in \mathbf{C} : |Imz| < 1/2\}$, which is bounded and continuous on its closure, $\forall t \in \mathbf{R}$.

(B3) *Adiabatic evolution.*

The perturbation is constant for $t < 0$, $V(t) \equiv V(0)$, and then *slowly* changes over a time interval τ such that $V^\tau(t) = V(s)$, where $s = t/\tau \in [0,1]$ is the rescaled time. We also assume that $V(s)$ is twice differentiable in $s \in [0,1]$ as a bounded operator, such that

$$\mathbf{R} \ni r \to \Delta^{ir} \partial_s^j V(s) \Delta^{-ir} \in \mathcal{M} , \quad j = 0,1,2 , \tag{57}$$

has an analytic continuation to the strip $\{z \in \mathbf{C} : |Imz| < 1/2\}$, which is bounded and continuous on its closure. This follows if we assume that there exists $\delta > 0$, independent of s and $i = 1, \ldots, n$, such that

$$e^{-\beta_i u/2} \partial_s^j \widetilde{f}_i(u,s) \in H^2(\delta, \mathcal{B}) , \tag{58}$$

the Hardy class of analytic functions, for $j = 0, 1, 2$. This assumption is needed to prove an adiabatic theorem for states close to NESS.[6]

Let $\widetilde{U}_g$ be the propagator generated by the adjoint of the C-Liouvillean, i.e.,

$$\partial_t \widetilde{U}_g(t,t') = -iL_g^*(t)\widetilde{U}_g(t,t'), \tag{59}$$

$$\widetilde{U}_g(t,t) = 1. \tag{60}$$

Assumption (B2) implies that the perturbation is bounded, and hence the domain of $L_g^{\#}$, where $L_g^{\#}$ stands for L_g or L_g^*, is

$$\mathcal{D}(L_g^{\#}) = \mathcal{D}(\mathcal{L}_0),$$

and $U_g, \widetilde{U}_g$ are bounded and strongly continuous in t and t'.

4. The C-Liouvillean and NESS

In [16, 21, 22], it is shown that, when the perturbation is time-independent, and under reasonable regularity assumptions on the form factors, the state of the coupled system converges to a nonequilibrium steady state (NESS) which is related to a zero-energy resonance of the adjoint of the C-Liouvillean. Here, we study the C-Liouvillean in the *time-dependent* case, and relate a zero-energy resonance to the *instantaneous* NESS. The statements made in this section have been proven in [16] (see also [14, 15]) for the *time-independent* case. Extending those results to the *time-dependent* case is *straightforward*, since we study the spectrum of the Liouvillean at each *fixed* moment of time. However, a sketch of the proofs of all the statements made in this section is given in the Appendix to make the presentation self-contained.

We first study the spectrum of L_g^* using complex spectral deformation techniques as developed in [12, 14–16].

Let $\mathbf{u}_i$ be the unitary transformation generating translations in energy for the i^{th} reservoir, $i = 1, \ldots, n$. More precisely, for $f_i \in L^2(\mathbf{R}; \mathcal{B})$,

$$\mathbf{u}_i(\theta) f_i(u) = f_i^{\theta}(u) = f_i(u+\theta).$$

Moreover, let

$$U_i(\theta) := \Gamma_i\big(\mathbf{u}_i(\theta)\big)$$

denote the second quantization of $\mathbf{u}_i(\theta)$.

Explicitly, $U_i(\theta) = e^{-i\theta A_i}$, where $A_i := id\Gamma_i(\partial_{u_i})$ is the second quantization of the generator of energy translations for the i^{th}reservoir, $i = 1, \ldots, n$. We set

$$U(\theta) := \mathbf{1}^{\Sigma} \otimes \mathbf{1}^{\Sigma} \otimes U_1(\theta) \otimes \cdots \otimes U_n(\theta). \tag{61}$$

[6]When the reservoirs are formed of nonrelativistic fermions in $\mathbf{R}^3$, an example of a form factor satisfying assumptions (B1)–(B3) is given by

$$f_i(u,s) = h_i(s)|u|^{1/4} e^{-|u|^2},$$

where $h_i(s)$ is twice differentiable in s.

Define

$$L_g^*(t,\theta) := U(\theta)L_g^*(t)U(-\theta)\,, \tag{62}$$

which is given by

$$L_g^*(t,\theta) = \mathcal{L}_0 + N\theta + g\widetilde{V}^{tot}(t,\theta)\,, \tag{63}$$

$\mathcal{L}_0 = \mathcal{L}^\Sigma + \sum_i \mathcal{L}^{\mathcal{R}_i}$, $\mathcal{L}^{\mathcal{R}_i} = d\Gamma(u_i), i = 1,\ldots,n$, $N = \sum_i N_i$, the total particle number operator, and

$$\widetilde{V}^{tot}(t,\theta) = \sum_i \Big\{ \sigma_1 \otimes \mathbf{1}^\Sigma \otimes \Big[a_i\big(f_{i,\beta_i}^{(\theta)}(t)\big) + a_i^*\big(f_{i,\beta_i}^{(\theta)}(t)\big)\Big] - i\mathbf{1}^\Sigma \otimes (\rho^\Sigma)^{-1/2}$$
$$\sigma_1(\rho^\Sigma)^{1/2} \otimes (-1)^{N_i}\Big[a_i\big(e^{\beta_i u_i/2} f_{i,\beta_i}^{\#(\theta)}(t)\big) + a_i^*\big(e^{-\beta_i u_i/2} f_{i,\beta_i}^{\#(\theta)}(t)\big)\Big]\Big\}\,.$$

It follows from assumption (B2) that, for $\theta \in I(\delta)$, $\widetilde{V}_g^{tot}(t,\theta)$ is a bounded operator. Hence $L_g^*(t,\theta)$ is well-defined and closed on the domain $\mathcal{D} := \mathcal{D}(N)\cap\mathcal{D}(\mathcal{L}^{\mathcal{R}_1})\cap\cdots\cap\mathcal{D}(\mathcal{L}^{\mathcal{R}_n})$. When the coupling $g = 0$, the pure point spectrum of $\mathcal{L}_0$ is $\sigma_{pp}(\mathcal{L}_0) = \{-2\omega_0, 0, 2\omega_0\}$, with double degeneracy at 0, and the continuous spectrum of $\mathcal{L}_0$ is $\sigma_{cont}(\mathcal{L}_0) = \mathbf{R}$. Let

$$\mathcal{L}_0(\theta) := \mathcal{L}_0 + N\theta\,.$$

We have the following two lemmas.

Lemma 4.1. *For $\theta \in \mathbf{C}$, the following holds.*

(i) *For any $\psi \in \mathcal{D}$, one has*

$$\|\mathcal{L}_0(\theta)\psi\|^2 = \|\mathcal{L}_0(Re\theta)\psi\|^2 + |Im\theta|^2\|N\psi\|^2\,. \tag{64}$$

(ii) *If $Im\theta \neq 0$, then $\mathcal{L}_0(\theta)$ is a normal operator satisfying*

$$\mathcal{L}_0(\theta)^* = \mathcal{L}_0(\overline{\theta})\,, \tag{65}$$

and $\mathcal{D}(\mathcal{L}_0(\theta)) = \mathcal{D}$.

(iii) *The spectrum of $\mathcal{L}_0(\theta)$ is*

$$\sigma_{cont}\big(\mathcal{L}_0(\theta)\big) = \{n\theta + s : n \in \mathbf{N}\backslash\{0\} \quad and \quad s \in \mathbf{R}\}\,, \tag{66}$$

$$\sigma_{pp}\big(\mathcal{L}_0(\theta)\big) = \{E_j : j = 0,\ldots,3\}\,, \tag{67}$$

where $E_{0,1} = 0, E_2 = -2\omega_0$ and $E_3 = 2\omega_0$, (the eigenvalues of $\mathcal{L}^\Sigma$).

Lemma 4.2. *Suppose assumptions* (B1) *and* (B2) *hold, and assume that $(g,\theta) \in \mathbf{C} \times I^-(\delta)$. Then, for each fixed time $t \in \mathbf{R}$, the following holds.*

(i) *$\mathcal{D}(L_g^*(t,\theta)) = \mathcal{D}$ and $(L_g^*(t,\theta))^* = L_{\overline{g}}(t,\overline{\theta})$.*

(ii) *The map $(g,\theta) \to L_g^*(t,\theta)$ from $\mathbf{C} \times I^-(\delta)$ to the set of closed operators on $\mathcal{H}$ is an analytic family* (*of type A*) *in each variable separately;* (*see* [17], *Chapter V, Section* 3.2).

(iii) *For $g \in \mathbf{R}$ finite and Imz large enough,*

$$s - \lim_{Im\theta\uparrow 0}\big(L_g^*(t,\theta) - z\big)^{-1} = \big(L_g^*(t, Re\theta) - z\big)^{-1}\,. \tag{68}$$

We now apply degenerate perturbation theory, as developed in [12], to compute the spectrum of $L_g^*(t,\theta)$. Using contour integration, one may define the projection onto the perturbed eigenstates of $L_g^*(t,\theta)$, for $\theta \in I^-(\delta)$. Let

$$P_g(t,\theta) := \oint_\gamma \frac{dz}{2\pi i}\big(z - L_g^*(t,\theta)\big)^{-1}\,, \tag{69}$$

where γ is a contour that encloses the eigenvalues $E_j, j = 0,\ldots,3$, at a distance $d > 0$, such that, for sufficiently small $|g|$ (to be specified below) the contour also encloses $E_j(g,t)$, the isolated eigenvalues of $L_g^*(t,\theta)$. We let

$$P_0 = \mathbf{1}^\Sigma \otimes \mathbf{1}^\Sigma \otimes |\widetilde{\Omega}^{\mathcal{R}_n} \otimes \cdots \otimes \widetilde{\Omega}^{\mathcal{R}_1}\rangle\langle\widetilde{\Omega}^{\mathcal{R}_1} \otimes \cdots \otimes \widetilde{\Omega}^{\mathcal{R}_n}|\,,$$

where $\mathbf{1}^\Sigma$ corresponds to the identity on $\mathcal{H}^\Sigma$ and $\widetilde{\Omega}^{\mathcal{R}_i}$ corresponds to the vacuum state in $\mathcal{F}^{\mathcal{R}_i}(L^2(\mathbf{R};\mathcal{B}))$. Furthermore, we define

$$T_g(t) := P_0 P_g(t,\theta) P_0\,. \tag{70}$$

Consider the isomorphism

$$S_g(t,\theta) := T_g^{-1/2}(t) P_0 P_g(t,\theta) : Ran\big(P_g(t,\theta)\big) \to Ran(P_0) \tag{71}$$

and its inverse[7]

$$S_g^{-1}(t,\theta) := P_g(t,\theta) P_0 T_g^{-1/2}(t) : Ran(P_0) \to Ran\big(P_g(t,\theta)\big)\,. \tag{72}$$

We set

$$M_g(t) := P_0 P_g(t,\theta) L_g^*(t,\theta) P_g(t,\theta) P_0\,, \tag{73}$$

and define the quasi-C-Liouvillean by

$$\Sigma_g(t) := S_g(t,\theta) P_g(t,\theta) L_g^*(t,\theta) P_g(t,\theta) S_g^{-1}(t,\theta) = T_g^{-1/2}(t) M_g(t) T_g^{-1/2}(t)\,. \tag{74}$$

Let $k = min\{\delta, \pi/\beta_1, \ldots, \pi/\beta_n\}$, where δ appears in assumption (B2), Section 3, and $\beta_1,\ldots,\beta_n$, are the inverse temperatures of the reservoirs $\mathcal{R}_1,\ldots,\mathcal{R}_n$, respectively. For $\theta \in I^-(k)$ (see (52)), we choose a parameter ν such that

$$-k < \nu < 0 \quad \text{and} \quad -k < Im\theta < -\frac{k+|\nu|}{2}\,. \tag{75}$$

We also choose a constant $g_1 > 0$ such that

$$g_1 C < (k - |\nu|)/2\,, \tag{76}$$

where

$$\begin{aligned} C &:= \sup_{\theta\in I(\delta), t\in\mathbf{R}} \|\widetilde{V}^{tot}(t,\theta)\| \qquad (77)\\ &\le \sup_{t\in\mathbf{R}, z\in I(\delta)} \frac{\sqrt{2}}{2} \sum_i |1 + e^{-\beta_i z}|^{-1/2} \Big(3\|\widetilde{f}_i(t)\|_{H^2(\delta,\mathcal{B})} + \|e^{-\beta_i u/2}\widetilde{f}_i\|_{H^2(\delta,\mathcal{B})}\Big)\,, \end{aligned}$$

which is finite due to assumption (B2).

[7] It follows from (78), Theorem 4.3 (i) below, that $T_g(t) \to 1$ on $Ran(P_g(t,\theta))$ as $g \to 0$, and hence $S_g(t,\theta)$ is a well-defined operator on $Ran(P_g(t,\theta))$. By (70), it has the right inverse $S_g^{-1}(t,\theta)$. Moreover, $dimRan(P_g(t,\theta)) = dimRan(P_0)$ for g small enough, and hence $S_g^{-1}(t,\theta)$ is the inverse of $S_g(t,\theta)$.

Theorem 4.3. *Suppose that assumptions* (B1) *and* (B2) *hold. Then, for $g_1 > 0$ satisfying* (76)*, $\theta \in I^-(k)$, ν satisfying* (75)*, and $t \in \mathbf{R}$ fixed, the following holds uniformly in t, i.e., g_1 is independent of t.*

(i) *If $|g| < g_1$, the essential spectrum of the operator $L_g^*(t,\theta)$ is contained in the half-plane $\mathbf{C}\backslash\Xi(\nu)$, where $\Xi(\nu) := \{z \in \mathbf{C} : Imz \geq \nu\}$. Moreover, the discrete spectrum of $L_g^*(t,\theta)$ is independent of $\theta \in I^-(k)$. If $|g| < 1/2g_1$, then the spectral projections $P_g(t,\theta)$, associated to the spectrum of $L_g^*(t,\theta)$ in the half-plane $\Xi(\nu)$, are analytic in g and satisfy the estimate*

$$\|P_g(t,\theta) - P_0\| < 1\,. \tag{78}$$

(ii) *If $|g| < g_1/2$, then the quasi-C-Liouvillean $\Sigma_g(t)$ defined in* (74) *depends analytically on g, and has a Taylor expansion*

$$\Sigma_g(t) = \mathcal{L}^\Sigma + \sum_{j=1}^{\infty} g^{2j}\Sigma^{(2j)}(t)\,. \tag{79}$$

The first non-trivial coefficient in (79) *is*

$$\Sigma^{(2)}(t) = \frac{1}{2}\oint_\gamma \frac{dz}{2\pi i}\Big(\xi(z,t)\big(z-\mathcal{L}^\Sigma\big)^{-1} + \big(z-\mathcal{L}^\Sigma\big)^{-1}\xi(z,t)\Big)\,,$$

where $\xi(z,t) := P_0\widetilde{V}^{tot}(t,\theta)(z-\mathcal{L}_0(\theta))^{-1}\widetilde{V}^{tot}(t,\theta)P_0$.

In fact, one may apply second order perturbation theory to calculate the perturbed eigenvalues of $L_g^*(t,\theta)$. To second order in the coupling g,

$$E_0(g,t) = 0\,,$$
$$E_1(g,t) = -i\pi g^2\sum_i \|\widetilde{f}_i(2\omega_0,t)\|^2_{\mathcal{B}} + O(g^4)\,,$$

and

$$E_{2,3}(g,t) = \mp\left(2\omega_0 - \frac{1}{2}g^2\mathcal{P}V\int_{\mathbf{R}} du\frac{1}{2\omega_0 - u}\sum_i\|\widetilde{f}_i(u,t)\|^2_{\mathcal{B}}\right) - i\frac{\pi}{2}g^2\sum_i\|\widetilde{f}_i(2\omega_0,t)\|^2_{\mathcal{B}} + O(g^4)\,,$$

where $\mathcal{P}V$ denotes the Cauchy principal value (see the Appendix).

The following corollary follows for the case of *time-independent* interactions; (see [16, 21, 22]).

Define

$$D := \mathbf{1}^\Sigma \otimes \mathbf{1}^\Sigma \otimes e^{-k\widetilde{A}_{\mathcal{R}_1}} \otimes \cdots \otimes e^{-k\widetilde{A}_{\mathcal{R}_n}}\,, \tag{80}$$

where $\widetilde{A}_{\mathcal{R}_i} := d\Gamma(\sqrt{p_i^2+1})$, and $p_i := i\partial_{u_i}$ is the generator of energy translations for $\mathcal{R}_i$, $i = 1,\ldots,n$. Note that D is a positive bounded operator on $\mathcal{H}$ such that $Ran(D)$ is dense in $\mathcal{H}$ and $D\Omega = \Omega$. This operator will act as a regulator which is used to apply complex deformation techniques. Let $\alpha_g^t \equiv \alpha_g^{t,0}$.

Corollary 4.4 (NESS). *Suppose assumptions* (B1) *and* (B2) *hold, and that the perturbation* $V(t) \equiv V$ *is time-independent. Then there exists* $g_1 > 0$ *such that, for* $0 < |g| < g_1$ *and* $a\Omega \in \mathcal{D}(D^{-1})$*, the following limit exists,*

$$\lim_{t\to\infty} \langle \Omega, \alpha_g^t(a)\Omega \rangle = \langle \Omega_g, D^{-1}a\Omega \rangle , \tag{81}$$

where Ω_g *corresponds to the zero-energy resonance of* L_g^**, and* α_g^t *is the perturbed dynamics. For* $a \in \mathcal{O}^{test}$*, a dense subset of* $\mathcal{O}$ *(that will be specified below), this limit is exponentially fast, with relaxation time* $\tau_R = O(g^{-2})$*.*[8]

Moreover, [16,21,22] prove strict positivity of entropy production in the latter case, which is consistent with Clausius' formulation of the second law of thermodynamics. See [10] for another proof using scattering theory of the convergence to a NESS and strict positivity of entropy production when two free fermionic reservoirs at different temperatures or chemical potentials are coupled.

5. Quasi-static evolution of NESS

In this section, we apply Theorem 2.2, Section 2, to investigate the quasi-static evolution of NESS of the model system introduced in Section 3.

Together with assumption (B1), we assume (B3), i.e., $V^\tau(t) = V(s)$, where $s \in [0,1]$ is the rescaled time with sufficient smoothness properties of the interaction. From Theorem 4.3, Section 4, we know the spectrum of the deformed adjoint of the C-Liouvillean, $L_g^*(t,\theta) = U(\theta)L_g^*(t)U(-\theta)$, for $\theta \in I^-(k)$, where $k = \min(\delta, \pi/\beta_1, \ldots, \pi/\beta_n)$, and δ appears in assumption (B3). Let γ_0 be a contour enclosing only the zero eigenvalue of $L_g^*(s,\theta)$, for all $s \in [0,1]$, and

$$P_g^0(s,\theta) := \oint_{\gamma_0} \frac{dz}{2\pi i} \big(z - L_g^*(s,\theta)\big)^{-1} , \tag{82}$$

the spectral projection onto the state corresponding to the zero eigenvalue of $L_g^*(s,\theta)$. Moreover, let $\mathbf{h}^{test} = \mathcal{D}(e^{k\sqrt{p^2+1}})$, and $\mathcal{O}^{\mathcal{R},test}$ be the algebra generated by $b^{\#}(f), f \in \mathbf{h}^{test}$, and $\mathbf{1}^{\mathcal{R}}$. Note that $\mathcal{O}^{\mathcal{R},test}$ is dense in $\mathcal{O}^{\mathcal{R}}$. Define

$$\mathcal{O}^{test} := \mathcal{O}^{\Sigma} \otimes \mathcal{O}^{\mathcal{R}_1,test} \otimes \cdots \mathcal{O}^{\mathcal{R}_n,test} , \tag{83}$$

which is dense in $\mathcal{O}$, and

$$\mathcal{C} := \big\{a\Omega : a \in \mathcal{O}^{test}\big\} \equiv \mathcal{D}\big(D^{-1}\big) ,$$

where D is the positive operator as defined in (80), Section 4. We make the following additional assumption.

(B4) The perturbation Hamiltonian $V(s) \in \mathcal{O}^{test}$, for $s \in [0,1]$.

[8] In fact, by assuming additional analyticity of the interacting Hamiltonian, one may show that this result holds for any initial state normal to ω; see [16, 21, 22].

In order to characterize the quasi-static evolution of nonequilibrium steady states, we introduce the new notion of an *instantaneous* NESS. Define an *instantaneous* NESS vector to be

$$\Omega_g(s) := DU(-\theta)P_g^0(s,\theta)U(\theta)D\Omega\,. \tag{84}$$

Note that Ω_g from Corollary 4.4, Section 4, has the same form as (84).

It is important to note that introducing the operator D is needed to remove the complex deformation.

We have the following Theorem, which effectively says that if a system, which is initially in a NESS, is perturbed slowly over a time scale $\tau \gg \tau_R$, where τ_R is some generic time scale ($\tau_R = \max_{s\in[0,1]} \tau_{R(s)}$, and $\tau_{R(s)}$ is the relaxation time to a NESS, see proof of Corollary 4.4 in the Appendix), then the real state of the system is infinitesimally close to the *instantaneous* NESS, and the difference of the two states is bounded from above by a term of order $O(\tau^{-1})$.

Theorem 5.1 (Adiabatic theorem for NESS). *Suppose assumptions* (B1)*,* (B3) *and* (B4) *hold. Then there exists* $g_1 > 0$*, independent of* $s \in [0,1]$*, such that, for* $a \in \mathcal{O}^{test}$*,* $s \in [0,1]$*, and* $0 < |g| < g_1$*, the following estimate holds*

$$\sup_{s\in[0,1]} \Big|\big\langle \Omega_g(0), D^{-1}\alpha_g^{\tau s}(a)\Omega\big\rangle - \big\langle \Omega_g(s), D^{-1}a\Omega\big\rangle\Big| = O(\tau^{-1})\,, \tag{85}$$

as $\tau \to \infty$*.*

Proof. Note that assumption (B3) implies assumption (B2), and hence the results of Theorem 4.3 about the spectrum of $L_g^*(t,\theta)$, for $\theta \in I^-(k)$ and *fixed* $t \in \mathbf{R}$, hold. The proof is now reduced to showing that the assumptions of Theorem 2.2 are satisfied. Choose $\theta \in I^-(k)$. It follows from assumption (B3) and Lemma A.1 in the Appendix, that the deformed C-Liouvillean $L_g^*(s,\theta)$ with common dense domain $\mathcal{D} = \mathcal{D}(\mathcal{L}_0) \cap \mathcal{D}(N)$ generates the propagator $\widetilde{U}_g^{(\tau)}(s,s',\theta), s' \le s$, which is given by

$$\partial_s \widetilde{U}_g^{(\tau)}(s,s',\theta) = -i\tau L_g^*(s,\theta)\widetilde{U}_g^{(\tau)}(s,s',\theta)\,, \quad \text{for} \quad s' \le s; \widetilde{U}_g^{(\tau)}(s,s,\theta) = 1\,. \tag{86}$$

This implies that (A1) and (A2) are satisfied. Furthermore, (A3) follows from the second resolvent identity

$$\big(L_g^*(s,\theta) - z\big)^{-1} = \big(\mathcal{L}_0(\theta) - z\big)^{-1}\Big(1 + g\widetilde{V}^{tot}(s,\theta)\big(\mathcal{L}_0(\theta) - z\big)^{-1}\Big)^{-1}\,, \tag{87}$$

and the results of Theorem 4.3, Section 4. We also know that zero is an isolated simple eigenvalue of $L_g^*(s,\theta)$ such that $dist(0, \sigma(L_g^*(s,\theta))\backslash\{0\}) > d$, where $d > 0$ is a constant independent of $s \in [0,1]$. This implies that assumption (A4) holds. Again using the resolvent equation (87) and assumption (B3), $P_g^0(s,\theta)$ defined in (82) is twice differentiable as a bounded operator for all $s \in [0,1]$, which imply (A5). Let $\widetilde{U}_a^{(\tau)}(s,s',\theta)$ (with domain $\mathcal{D}$) be the propagator of the *deformed* adiabatic evolution given by

$$\partial_s \widetilde{U}_a^{(\tau)}(s,s',\theta) = -i\tau L_a^*(s,\theta)\widetilde{U}_a^{(\tau)}(s,s',\theta) \quad \text{for} \quad s' \le s\,; \widetilde{U}_a^{(\tau)}(s,s,\theta) = 1\,, \tag{88}$$

and

$$L_a^*(s,\theta) = L_g^*(s,\theta) + \frac{i}{\tau}\left[\dot{P}_g(s,\theta), P_g(s,\theta)\right]. \tag{89}$$

(Here, the $\dot{()}$ stands for differentiation with respect to s.) Since (A1)–(A5) are satisfied, the results of Theorem 2.2 hold.

$$P_g^0(s,\theta)\widetilde{U}_a^{(\tau)}(s,0,\theta) = \widetilde{U}_a^{(\tau)}(s,0,\theta)P_g^0(0,\theta), \tag{90}$$

and

$$\sup_{s\in[0,1]} \|\widetilde{U}_g^{(\tau)}(s,0,\theta) - \widetilde{U}_a^{(\tau)}(s,0,\theta)\| = O(\tau^{-1}), \tag{91}$$

as $\tau \to \infty$.

For h the single particle Hamiltonian of the free fermions, e^{iht} leaves $D(e^{k\sqrt{p^2+1}})$ invariant. Therefore, for $a \in \mathcal{O}^{test}$, $\alpha_0^t(a) \in \mathcal{O}^{test}$, where α_0^t corresponds to the free time evolution. Moreover, together with assumption (B4) and the boundedness of V, this implies (using a Dyson series expansion) that $\alpha_g^{\tau s}(a) \in \mathcal{O}^{test}$.

Now, applying the time evolution on $C(\mathcal{O},\Omega)$, and remembering that $D\Omega = \Omega$, $U(\theta)\Omega = \Omega$, the fact that $U(\theta)$ and D commute, and the definition of the instantaneous NESS, it follows that

$$\left\langle \Omega_g(0), D^{-1}\alpha_g^{\tau s}(a)\Omega \right\rangle = \left\langle \widetilde{U}_g^{(\tau)}(s,0,\theta)P_g^0(0,\theta)\Omega, a(\overline{\theta})\Omega \right\rangle. \tag{92}$$

Using the results of Theorem 2.2, it follows that

$$\begin{aligned}
\left\langle \widetilde{U}_g^{(\tau)}(s,0,\theta)P_g^0(0,\theta)\Omega, a(\overline{\theta})\Omega \right\rangle &= \left\langle \widetilde{U}_a^{(\tau)}(s,0,\theta)P_g^0(0,\theta)\Omega, a(\overline{\theta})\Omega \right\rangle + O(\tau^{-1}) \\
&= \left\langle P_g^0(s,\theta)\widetilde{U}_a^{(\tau)}(s,0,\theta)\Omega, a(\overline{\theta})\Omega \right\rangle + O(\tau^{-1}) \\
&= \left\langle P_g^0(s,\theta)\widetilde{U}_g^{(\tau)}(s,0,\theta)\Omega, a(\overline{\theta})\Omega \right\rangle + O(\tau^{-1}).
\end{aligned}$$

The fact that $(\widetilde{U}_g^{(\tau)}(s,0,\theta))^*\Omega = \Omega$ implies

$$\begin{aligned}
DP_g^0(s,\theta)\widetilde{U}_g^{(\tau)}(s,0,\theta) &= |\Omega_g(s,\theta)\rangle\langle\Omega|\widetilde{U}_g^{(\tau)}(s,0,\theta) \\
&= |\Omega_g(s,\theta)\rangle\langle(\widetilde{U}_g^{(\tau)}(s,0,\theta))^*\Omega| \\
&= |\Omega_g(s,\theta)\rangle\langle\Omega| = DP_g^0(s,\theta).
\end{aligned}$$

It follows that

$$\left\langle \Omega_g(0), D^{-1}\alpha_g^{(\tau s)}(a)\Omega \right\rangle = \left\langle \Omega_g(s), D^{-1}a\Omega \right\rangle + O(\tau^{-1}),$$

for large τ. □

Remarks. (1) *Positivity of entropy production.* If the interaction Hamiltonian $gV(t)$ is time-periodic with period τ, i.e., $V(t+\tau) = V(t)$, it is shown in [3] that the final state of the coupled system (introduced in Section 3) converges to a time periodic state after very many periods. It is also shown that entropy production per cycle is strictly positive (Theorem 6.3 in [3]). The infinite period limit, $\tau \to \infty$, is equivalent to the quasi-static limit. Hence, entropy production in the quasi-static evolution of NESS of the model considered in this paper is strictly positive.

(2) *An example of a reversible isothermal process.* As a second application of Theorem 2.2 in quantum statistical mechanics, one may consider a concrete example of an isothermal process of a small system coupled to a *single* fermionic reservoir, and calculate an explicit rate of convergence ($O(\tau^{-1})$) between the instantaneous equilibrium state and the true state of the system in the quasi-static limit $\tau \to \infty$ (see [1]). Under suitable assumptions on the form factors, one may show that there exists a constant $g_1 > 0$ such that, for a in a dense subset of $\mathcal{O}$ and $0 < |g| < g_1$, the following estimate holds

$$\left|\rho_{\tau s}(a) - \omega^{\beta}_{\tau s}(a)\right| = O(\tau^{-1}) \,, \tag{93}$$

as $\tau \to \infty$, where $\rho_{\tau s}$ is the true state of the system at time $t = \tau s$, and $\omega^{\beta}_{\tau s}$ is the instantaneous equilibrium state, which corresponds to the zero eigenvalue of the time-dependent *standard* Liouvillean.

Appendix A.

Existence of the *deformed* **time evolution**

Choose $\theta \in I^-(\delta)$, where δ appears in assumption (B2), Section 3. The *deformed* time evolution is given by the propagator $\widetilde{U}_g(t,t',\theta)$ which satisfies

$$\partial_t \widetilde{U}_g(t,t',\theta) = -iL^*_g(t,\theta)\widetilde{U}_g(t,t',\theta) \,, \quad \widetilde{U}_g(t,t,\theta) = 1 \,.$$

The following lemma guarantees the existence of $\widetilde{U}_g(t,t',\theta)$. Let

$$\mathcal{D} := \mathcal{D}(\mathcal{L}_0) \cap \mathcal{D}(N) \,,$$

and denote by

$$\begin{aligned} C &:= \sup_{t\in\mathbf{R}} \sup_{\theta\in I^-(\delta)} \|\widetilde{V}^{tot}(t,\theta)\| \\ &\leq \frac{\sqrt{2}}{2} \sup_{t\in\mathbf{R}, z\in I(\delta)} \sum_i \left|1 + e^{-\beta_i z}\right|^{-1/2} \Big(3\|\widetilde{f}_i(t)\|_{H^2(\delta,\mathcal{B})} \\ &\quad + \|e^{-\beta_i u_i/2}\widetilde{f}_i(t)\|_{H^2(\delta,\mathcal{B})}\Big) < \infty \end{aligned}$$

due to assumption (B2), Section 3.

Lemma A.1. *Assume* (B2)*, choose $\theta \in I^-(\delta) \cup \mathbf{R}$ and $|g| < g_1$, and fix $t \in \mathbf{R}$. Then*

(i) *$L^*_g(t,\theta)$ with domain $\mathcal{D}$ generates a contraction semi-group $e^{-i\sigma L^*_g(t,\theta)}, \sigma \geq 0$ on $\mathcal{H}$.*

(ii) *For $\psi \in \mathcal{D}$, $e^{-i\sigma L^*_g(t,\theta)}\psi$ is analytic in $\theta \in I^-(\delta)$. For $\theta' \in \mathbf{R}$ and $\theta \in I^-(\delta) \cup \mathbf{R}$,*

$$U(\theta')e^{-i\sigma L^*_g(t,\theta)}U(-\theta') = e^{-i\sigma L^*_g(t,\theta+\theta')} \,.$$

(iii) $\widetilde{U}_g(t,t',\theta)\widetilde{U}_g(t',t'',\theta) = \widetilde{U}_g(t,t'',\theta)$ *for* $t \geq t' \geq t''$.

(iv) $\widetilde{U}_g(t,t',\theta)\mathcal{D} \subset \mathcal{D}$, *and for* $\psi \in \mathcal{D}$, $\widetilde{U}_g(t,t',\theta)\psi$ *is differentiable in* t *and* t' *such that*

$$\partial_t \widetilde{U}_g(t,t',\theta)\psi = -iL_g^*(t,\theta)\widetilde{U}_g(t,t',\theta)\psi\,,$$
$$\partial_{t'} \widetilde{U}_g(t,t',\theta)\psi = i\widetilde{U}_g(t,t',\theta)L_g^*(t',\theta)\psi\,.$$

(v) *For* $\theta' \in \mathbf{R}$,

$$U(\theta')\widetilde{U}_g(t,t',\theta)U(-\theta') = \widetilde{U}_g(t,t',\theta+\theta')\,.$$

Moreover, $\widetilde{U}_g(t,t',\theta)$ *is analytic in* $\theta \in I^-(\delta)$.

Proof. Claim (i) follows from Phillip's Theorem for the perturbation of semigroups (see [17] Chapter IX). Claim (ii) follows from assumption (B2), the resolvent identity

$$\big(L_g^*(t,\theta)-z\big)^{-1} = \big(\mathcal{L}_0(\theta)-z\big)^{-1}\Big(1+\widetilde{V}^{tot}(t,\theta)\big(\mathcal{L}_0(\theta)-z\big)^{-1}\Big)^{-1}\,,$$
$$U(\theta')L_g^*(t,\theta)U(-\theta') = L_g^*(t,\theta+\theta')\,,$$

and the fact that

$$e^{-i\sigma L_g^*(t,\theta)} = \frac{1}{2\pi i}\int_\Gamma e^{-\sigma z}\big(iL_g^*(t,\theta)-z\big)^{-1}dz\,,$$

where Γ is a contour encircling the spectrum of $L_g^*(t,\theta)$.

Claims (iii) and (iv) are consequences of Kato's Theorem [18], to which we refer the reader. Without loss of generality, rescale time such that $t=\tau s$, $s\in[0,1]$, and let $L_g^{*n}(s\tau,\theta) = L_g^*(\tau\frac{k}{n},\theta)$ for $n \in \mathbf{N}\backslash\{0\}$ and $s \in [\frac{k}{n},\frac{k+1}{n}], k=0,\dots,n-1$. Moreover, define $\widetilde{U}_g^n(\tau s,\tau s',\theta) := e^{-i\tau(s-s')L_g^{*n}(\tau\frac{k}{n},\theta)}$ if $\frac{k}{n} \leq s' \leq s \leq \frac{k+1}{n}$, and $\widetilde{U}_g^n(\tau s,\tau s',\theta) = \widetilde{U}_g^n(\tau s,\tau s'',\theta)\widetilde{U}_g^n(\tau s'',\tau s',\theta)$ if $0 \leq s' \leq s'' \leq s \leq 1$. It follows from (ii) for $\theta' \in \mathbf{R}$, that

$$U(\theta')\widetilde{U}_g^n(\tau s,\tau s',\theta)U(-\theta') = \widetilde{U}_g^n(\tau s,\tau s',\theta+\theta')\,,$$

and that $\widetilde{U}_g^n(\tau s,\tau s',\theta)$ is analytic in $\theta \in I^-(\delta)$, where δ appears in (B2). Claim (v) follows by taking the $n\to\infty$ limit (in norm). □

Glued Hilbert space representation

We want to show that

$$\mathcal{F}\big(L^2(\mathbf{R}^+;\mathcal{B})\big)\otimes\mathcal{F}\big(L^2(\mathbf{R}^+;\mathcal{B})\big) \cong \mathcal{F}\big(L^2(\mathbf{R};\mathcal{B})\big)\,.$$

Let Ω be the vacuum state in the fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$. For fermionic creation/annihilation operators on $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$,

$$b^{\#}(f) := \int m(u)dud\sigma f(u,\sigma)b^{\#}(u,\sigma)\,,\quad f\in L^2(\mathbf{R}^+;\mathcal{B})\,,$$

define the creation/annihilation operators on $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\otimes\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$ as

$$b_l^{\#}(f) := b^{\#}(f)\otimes \mathbf{1}\,;$$
$$b_r^{\#}(f) := (-1)^N \otimes b^{\#}(\overline{f})\,,$$

where $\bar{\cdot}$ corresponds to complex conjugation. Note that b_l and b_r anti-commute. Let $\widetilde{a}$ and $\widetilde{a}^*$ be the annihilation and creation operators on the fermionic Fock space $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$, such that they satisfy the usual CAR, and let $\widetilde{\Omega}$ be the vacuum state in $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$. An isomorphism between $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))\otimes\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B}))$ and $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$ follows by the identification

$$b_l^{\#}(f) \cong \widetilde{a}^{\#}((f,0))\,,$$
$$b_r^{\#}(g) \cong \widetilde{a}^{\#}((0,g))\,,$$
$$\Omega\otimes\Omega \cong \widetilde{\Omega}\,.$$

Now we claim that $\mathcal{F}(L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}))$ is isomorphic to $\mathcal{F}(L^2(\mathbf{R};\mathcal{B}))$. Consider the mapping

$$j : L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B}) \ni (f,g) \to h \in L^2(\mathbf{R};\mathcal{B})\,,$$

such that

$$h(u,\sigma) := \begin{cases} \sqrt{m(u)}f(u,\sigma)\,, & u \geq 0 \\ \sqrt{m(|u|)}g(|u|,\sigma)\,, & u < 0 \end{cases}.$$

This mapping is an isometry, since

$$\begin{aligned}\|h\|^2_{L^2(\mathbf{R};\mathcal{B})} &= \|(f,g)\|^2_{L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B})} \\ &= \int_{\mathbf{R}^+;\mathcal{B}} du d\sigma m(u)|f(u,\sigma)|^2 + \int_{\mathbf{R}^+;\mathcal{B}} du d\sigma m(u)|g(u,\sigma)|^2 \\ &= \|f\|^2_{L^2(\mathbf{R}^+;\mathcal{B})} + \|g\|^2_{L^2(\mathbf{R}^+;\mathcal{B})}\,.\end{aligned}$$

Moreover, the mapping j is an isomorphism, since, for given $h\in L^2(\mathbf{R};\mathcal{B})$, there exists a mapping $j^{-1} : h \to (f,g) \in L^2(\mathbf{R}^+;\mathcal{B})\oplus L^2(\mathbf{R}^+;\mathcal{B})$, such that

$$f(u,\sigma) := \frac{1}{\sqrt{m(u)}}h(u,\sigma)\,,\quad u>0\,,$$
$$g(u,\sigma) := \frac{1}{\sqrt{m(|u|)}}h(|u|,\sigma)\,,\quad u<0\,.$$

Proof of statements in Section 4[9]

Proof of Lemma 4.1. $\mathcal{L}_0(\theta)$ restricted to the $N = n\mathbf{1}$ sector is

$$\mathcal{L}_0^{(n)}(\theta) = \mathcal{L}^{\Sigma} + s_1 + \cdots + s_n + n\theta\,, \tag{94}$$

[9] Although the results in this subsection are a very simple extension of those proven in [14–16] to the time-dependent case, they are sketched here so that the presentation is self-contained. The reader can refer to those references for additional details.

where $s_1, \ldots, s_n$ are interpreted as one-particle multiplication operators. For $Im\theta \neq 0$, it also follows from (94) that

$$\mathcal{D} = \left\{ \psi = \{\psi^{(n)}\} : \psi^{(n)} \in \mathcal{D}\big(\mathcal{L}_0^{(n)}(\theta)\big) \text{ and } \sum_n \|\mathcal{L}_0^{(n)}(\theta)\psi^{(n)}\|^2 < \infty \right\},$$

and hence $\mathcal{L}_0(\theta)$ is a closed normal operator on $\mathcal{D}$. Claims (ii) and (iii) follow from the corresponding statements on the sector $N = n\mathbf{1}$. □

Proof of Lemma 4.2. The first claim (i) follows from the fact that $g\widetilde{V}^{tot}(t,\theta)$ is bounded for $\theta \in I(\delta)$ due to assumption (B2) and the fact that the reservoirs are fermionic. It also follows from assumption (B2) that $(g,\theta) \to L_g^*(t,\theta)$ is analytic in $\theta \in I^-(\delta)$. Analyticity in g is obvious from (62). Assume that $Re\theta = 0$. It follows from assumption (B2) that the resolvent formula

$$\big(L_g^*(t,\theta) - z\big)^{-1} = \big(\mathcal{L}_0(t,\theta) - z\big)^{-1}\Big(1 + g\widetilde{V}^{tot}(t,\theta)\big(\mathcal{L}_0(\theta) - z\big)^{-1}\Big)^{-1}, \qquad (95)$$

holds for small g, as long as z belongs to the half-plane $\{z \in \mathbf{C} : 0 < c < Imz\}$. Since $(\mathcal{L}_0(t,\theta) - z)^{-1}$ is uniformly bounded as $Im\theta \uparrow 0$ for $g \in \mathbf{R}$ and Imz large enough, and $\widetilde{V}^{tot}(t,\theta)$ is bounded and analytic in θ, claim (iii) follows from the Neumann series expansion of the resolvent of $L_g^*(t,\theta)$. □

Proof of Theorem 4.3. (i) The resolvent formula

$$\big(L_g^*(t,\theta) - z\big)^{-1} = \big(\mathcal{L}_0(\theta) - z\big)^{-1}\Big(1 + g\widetilde{V}^{tot}(t,\theta)\big(\mathcal{L}_0(\theta) - z\big)^{-1}\Big)^{-1}, \qquad (96)$$

holds for small g and z in the half-plane $\{z \in \mathbf{C} : 0 < c < Imz\}$. Note that

$$\begin{aligned}\big\|g\widetilde{V}^{tot}(t,\theta)\big(\mathcal{L}_0(\theta) - z\big)^{-1}\big\| &\leq |g|C\big\|\big(\mathcal{L}_0(\theta) - z\big)^{-1}\big\| \\ &\leq |g|C\frac{1}{dist\big(z, \eta(\mathcal{L}_0(\theta))\big)},\end{aligned}$$

where C is given by (77) and $\eta(\mathcal{L}_0(\theta))$ is the closure of the numerical range of $\mathcal{L}_0$. Fix g_1 such that it satisfies (76), and choose ϵ such that $\epsilon > \frac{k-|\nu|}{2} > 0$. Let

$$G(\nu,\epsilon) := \Big\{z \in \mathbf{C} : Imz > \nu; dist\Big(z, \eta\big(\mathcal{L}_0(\theta)\big)\Big) > \epsilon\Big\}.$$

Then

$$\sup_{z\in G(\nu,\epsilon)} \big\|g\widetilde{V}^{tot}(t,\theta)\big(\mathcal{L}_0(\theta) - z\big)^{-1}\big\| \leq \frac{|g|}{g_1},$$

uniformly in t. If $|g| < g_1$, the resolvent formula (96) holds on $G(\nu,\epsilon)$, and, for $m \geq 1$,

$$\sup_{z\in G(\nu,\epsilon)} \left\| \big(z - L_g^*(t,\theta)\big)^{-1} - \sum_{j=0}^{m-1} \big(z-\mathcal{L}_0(t,\theta)\big)^{-1}\Big(g\widetilde{V}^{tot}(t,\theta)\big(z-\mathcal{L}_0(\theta)\big)^{-1}\Big)^j \right\| \leq \frac{\left(\frac{|g|}{g_1}\right)^m}{1-\frac{|g|}{g_1}}\,, \tag{97}$$

uniformly in t. It follows that

$$\bigcup_{\epsilon > \frac{k-|\nu|}{2}} G(\nu,\epsilon) \subset \rho\big(L_g^*(t,\theta)\big)\,, \tag{98}$$

where $\rho(L_g^*(t,\theta))$ is the resolvent set of $L_g^*(t,\theta)$. Moreover, setting $m=1$ in (97), it follows that, for $|g| < g_1/2$,

$$\|P_g(t,\theta) - P_0\| < 1\,,$$

and hence $P_g(t,\theta)$ is analytic in g.

Fix $(g_0,\theta_0) \in \mathbf{C} \times I^-(\delta)$ such that $|g_0| < g_1$. Since $L_{g_0}^*(t,\theta_0)$ and $L_{g_0}^*(t,\theta)$ are unitarily equivalent if $(\theta-\theta_0) \in \mathbf{R}$ and the discrete eigenvalues of $L_{g_0}^*(t,\theta)$ are analytic functions with at most algebraic singularities in the neighbourhood of θ_0, it follows that the pure point spectrum of $L_{g_0}^*(t,\theta)$ is independent of θ.

(ii) Analyticity of $T_g(t)$ in g follows directly from (i) and the definition of $T_g(t)$. Since $\|T_g(t)-1\| < 1$ for $|g| < g_1/2$, $T_g^{-1/2}(t)$ is also analytic in g. Inserting the Neumann series for the resolvent of $L_g^*(t,\theta)$, gives

$$T_g(t) = 1 + \sum_{j=1}^{\infty} g^j T^{(j)}(t)\,, \tag{99}$$

with

$$T^{(j)}(t) = \oint_\gamma \frac{dz}{2\pi i}\big(z-\mathcal{L}^\Sigma\big)^{-1} P_0 \widetilde{V}^{tot}(t,\theta)((z-\mathcal{L}_0(\theta))^{-1}\widetilde{V}^{tot}(t,\theta))^{j-1} P_0 \big(z-\mathcal{L}^\Sigma\big)^{-1}\,. \tag{100}$$

Similarly,

$$M_g(t) = \mathcal{L}^\Sigma + \sum_{j=1}^{\infty} g^j M^{(j)}(t)\,, \tag{101}$$

with

$$M^{(j)}(t) = \oint_\gamma \frac{dz}{2\pi i} z\big(z-\mathcal{L}^\Sigma\big)^{-1} P_0 \widetilde{V}^{tot}(t,\theta)((z-\mathcal{L}_0(\theta))^{-1}\widetilde{V}^{tot}(t,\theta))^{j-1} P_0 \big(z-\mathcal{L}^\Sigma\big)^{-1}\,. \tag{102}$$

The odd terms in the above two expansions are zero due to the fact that P_0 projects onto the $N=0$ sector. The first non-trivial coefficient in the Taylor series of $\Sigma_g(t)$

is

$$\Sigma^{(2)}(t) = M^{(2)}(t) - \frac{1}{2}\big(T^{(2)}(t)\mathcal{L}^{\Sigma} + \mathcal{L}^{\Sigma}T^{(2)}(t)\big) \tag{103}$$

$$= \frac{1}{2}\oint_{\gamma}\frac{dz}{2\pi i}\Big(\xi(z,t)\big(z-\mathcal{L}^{\Sigma}\big)^{-1} + \big(z-\mathcal{L}^{\Sigma}\big)^{-1}\xi(z,t)\Big)\,, \tag{104}$$

with

$$\xi(z,t) = P_0\widetilde{V}_g^{tot}(t,\theta)\big(z-\mathcal{L}_0(\theta)\big)^{-1}\widetilde{V}_g^{tot}(t,\theta)P_0\,. \qquad \square$$

Details of the calculation of the discrete spectrum of $L_g^*(t,\theta)$. Denote by $P_k, k = 0,\ldots,3$, the spectral projection onto the eigenstates of $\mathcal{L}^{\Sigma}$, and let

$$\Gamma_k^{(2)}(t) := P_k\Sigma^{(2)}(t)P_k\,, \quad k = 0,\ldots,3\,.$$

Consider first the nondegenerate eigenvalues ($E_k = \mp 2\omega_0, k = 2,3$). Using the fact that

$$\lim_{\epsilon\searrow 0} Re\frac{1}{x-i\epsilon} = \mathcal{P}V\frac{1}{x}\,;$$

$$\lim_{\epsilon\searrow 0} Im\frac{1}{x-i\epsilon} = i\pi\delta(x)\,,$$

and applying the Cauchy integration formula gives

$$Re\Gamma_3^{(2)} = \frac{1}{2}\sum_i \mathcal{P}V\int_{\mathbf{R}} du\frac{\|\widetilde{f}_i(u,t)\|_{\mathcal{B}}^2}{u-2\omega_0}\,,$$

$$Im\Gamma_3^{(2)} = -\frac{\pi}{2}\sum_i \|\widetilde{f}_i(2\omega_0,t)\|_{\mathcal{B}}^2\,,$$

and

$$Re\Gamma_2^{(2)} = -\frac{1}{2}\sum_i \mathcal{P}V\int_{\mathbf{R}} du\frac{\|\widetilde{f}_i(u,t)\|_{\mathcal{B}}^2}{u-2\omega_0}\,,$$

$$Im\Gamma_2^{(2)} = -\frac{\pi}{2}\sum_i \|\widetilde{f}_i(2\omega_0,t)\|_{\mathcal{B}}^2\,.$$

Now apply degenerate perturbation theory for the zero eigenvalue. Using the definition of f_{i,β_i} and $f_{i,\beta_i}^{\#}$ given in Section 3,

$$Re\Gamma_{0,1}^{(2)} = 0\,,$$

$$Im\Gamma_{0,1}^{(2)} = -\pi\sum_i \frac{\|\widetilde{f}_i(2\omega_0,t)\|_{\mathcal{B}}^2}{cosh(\beta_i\omega_0)}\begin{pmatrix} e^{\beta_i\omega_0} & -e^{\beta_i\omega_0} \\ -e^{-\beta_i\omega_0} & e^{-\beta_i\omega_0}\end{pmatrix}.$$

Therefore, to second order in the coupling g,

$$E_{2,3}(g,t) = \mp \left(2\omega_0 - \frac{1}{2} g^2 \mathcal{P}V \int_{\mathbf{R}} du \frac{1}{2\omega_0 - u} \sum_i \|\widetilde{f}_i(u,t)\|^2_{\mathcal{B}} \right) - i\frac{\pi}{2} g^2 \sum_i \|\widetilde{f}_i(2\omega_0, t)\|^2_{\mathcal{B}} + O(g^4)\,,$$

while

$$E_{0,1}(g,t) = g^2 a_{0,1}(t) + O(g^4)\,,$$

where $a_{0,1}(t)$ are the eigenvalues of the matrix

$$-i\pi \sum_i \frac{\|\widetilde{f}_i(2\omega_0,t)\|^2_{\mathcal{B}}}{2\cosh(\beta_i\omega_0)} \begin{pmatrix} e^{\beta_i\omega_0} & -e^{\beta_i\omega_0} \\ -e^{-\beta_i\omega_0} & e^{-\beta_i\omega_0} \end{pmatrix}.$$

Since Ω is an eigenvector corresponding to the isolated zero eigenvalue of $L_g(t,\theta)$ (by construction, $L_g(t,\theta)\Omega = 0$), then zero is also an isolated eigenvalue of $L_g^*(t,\theta)$. (One way of seeing this is to take the adjoint of the spectral projection of $L_g(t,\theta)$ corresponding to Ω, which is defined using contour integration.) Note that $\psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to the zero eigenvalue of $\Sigma^{(2)}(t)$. Hence,

$$\begin{aligned} E_0(g,t) &= 0\,, \\ E_1(g,t) &= -i\pi g^2 \sum_i \|\widetilde{f}_i(2\omega_0,t)\|^2_{\mathcal{B}} + O(g^4)\,. \end{aligned}$$

Proof of Corollary 4.4 (*NESS*). Define $k := \min(\pi/\beta_1, \ldots, \pi/\beta_n, \delta)$, where δ appears in assumption (B2), and let $\theta \in I^-(k)$. We already know the spectrum of $L_g^*(t,\theta)$ from Theorem 4.3. For $a \in \mathcal{O}^{test}$,

$$\begin{aligned} \lim_{t\to\infty} \langle \Omega, \alpha_g^t(a)\Omega\rangle &= \lim_{t\to\infty} \langle \Omega, e^{itL_g} a e^{-itL_g}\Omega\rangle \\ &= \lim_{t\to\infty} \langle e^{-itL_g^*}\Omega, a\Omega\rangle \\ &= \lim_{t\to\infty} \langle e^{-itL_g^*(\theta)}\Omega, a(\overline{\theta})\Omega\rangle \\ &= \lim_{t\to\infty} \frac{1}{2\pi i} \left\langle \int_{-\infty}^{\infty} du \big(u + i\eta - L_g^*(\theta)\big)^{-1} e^{-i(u+i\eta)t}\Omega, a(\overline{\theta})\Omega \right\rangle, \end{aligned}$$

for $\eta > 0$. One may decompose the last integral into two parts (see for example [14]). The first part is

$$\lim_{t\to\infty} \frac{1}{2\pi i} \left\langle \oint_\gamma dz \big(z - L_g^*(\theta)\big)^{-1} e^{-izt}\Omega, a(\overline{\theta})\Omega \right\rangle = \langle \Omega_g, D^{-1} a\Omega\rangle\,,$$

where the zero-energy resonance is

$$\Omega_g := DU(-\theta)P_g^0(\theta)U(\theta)D\Omega = DU(-\theta)P_g^0(\theta)\Omega\,.$$

The second term converges to zero exponentially fast as $t \to \infty$, since

$$\frac{1}{2\pi i}\left\langle \int_{-\infty}^{\infty} \big(u - i(\mu-\epsilon) - L_g^*(\theta)\big)^{-1} e^{-i(u-i(\mu-\epsilon))t}\Omega, a(\overline{\theta})\Omega \right\rangle = O\big(e^{-(\mu-\epsilon')t}\big)\,,$$

where $0 < \epsilon' < \epsilon < |Im\theta| =: \mu$; (see also Theorem 19.2 in [25]). □

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